Chapter 2 Self Test

As the chapter notes, solving back-of-the-envelope problems requires a logical approach coupled with a dose of self-confidence. Such confidence comes with practice.

With all the hoopla over killer asteroids and comets these days, you might be asked by one of your less scientifically literate relatives the following question: "What are the odds of my house being struck by a meteor during my lifetime?" If I gave you the one critical piece of information you might well not know – that there are about 100,000 objects a year softball-size or larger that strike the Earth – could you quickly calculate a rough estimate?

If the answer is yes, you have probably mastered this habit of mind and you need read no further. If you haven't a clue how to proceed, or won't have any confidence in the answer you might obtain, read on.

A logical approach to these (or any other) problems that I have found effective is given here:

I. Understanding the Problem

a) *Write down* what you are looking for. It is absolutely essential to know where you are going if you hope to get there; include units. The units are critical as they clarify exactly what you want to know, and they will provide an extremely useful check that you have done the problem correctly.

b) Write down, in the simplest possible form, all the information you are given. Again, include units.

c) Draw a picture if appropriate - this often helps to visualize what you have, and what you need, to get an answer.

II. Identifying Additional Information

a) Write down things you may need to know and the units you wish to know them in.

b) Write down, in mathematical symbols if appropriate, the definitions of key words and phrases.

c) Don't get sidetracked by peripheral issues or minor possible corrections; remember, the point here is to get a *rough* estimate of some quantity.

d) Be explicit about the assumptions you are making.

III. Making Good Estimates

a) Estimate the things you don't know. This is obviously a key point and may not be easy at first. One trick is to start with either absurdly large or absurdly small values which you *know* must be wrong, and try to converge from these outrageous values to something more reasonable.

b) Think about whether or not your assumptions are justifiable.

IV. Getting the Answer

a) Calculate, keeping careful track of units and factors of ten; convert units when necessary (see Ch. 1).

b) Do a sanity check on your answer. This is not always straightforward - after all, if you knew roughly what the answer should be, you wouldn't have had to do the calculation! But you can at least check that the units of the answer are right (e.g., if you were trying to estimate the number of people doing something and got an answer in gallons, there's probably a problem). You might also have at least some clue about the size of the answer - e.g., if you were trying to estimate the number of high school teachers in New York City and got a number larger than 6.5 billion (the world's population) you can tell there's a problem.

Here are two worked examples, followed by a half dozen for you to try. The answers can be found at the end.

Sample problem 1: In March 2006, the NYC subway system set a 50-year record for the number of riders, averaging 4.9 million per day. Roughly how many gallons of gas were saved by the subway system that month (the power for the subway is largely hydroelectric power from Quebec)?

I. Understanding the Problem

a) What we want to know: gallons of gas saved in a month = gallons/month

b) What we are given: passengers per day: 4.9×10^6; month = March = 31 days

c) Picture probably not required.

II. Identifying Additional Information

a) What we need to know: average trip length per rider; number of passengers per car if they were driving instead; miles per gallon the average car uses

b) Definitions: all pretty straightforward

c) There are probably as many peripheral issues that could present stumbling blocks as there are people who attempt this problem. For example, "What about all those tour groups of 30 people who would otherwise all be on one bus?" What fraction of subway riders do you think they make up? I'd guess less than 1%, and we certainly aren't worried about corrections at the 1% level.

III. Making Good Estimates

a) average trip length: Something you will come to learn if you don't know it yet is that there are twenty NYC blocks per mile. So a ride from Columbia to Greenwich Village is a little over five miles. A commute from the outer boroughs can be more than ten; a short trip to the Metropolitan Opera is only 2.5. My estimate is 4 miles/trip. A factor of 10 smaller would be under a mile – most New Yorkers would just walk. A factor of ten longer would be 40 miles which is more than the distance from the northern Bronx to southern Brooklyn (which I suspect few people do regularly). Note how testing much bigger and much smaller numbers gives me added confidence in my estimate.

b) number of passengers per car: I drive as little as possible (which is why I live in NYC), but when I do, it looks to me as though the majority of people are alone in their cars. Perhaps the subways have a few couples or family groups traveling together, but I suspect the majority of trips are for commuting, and that is likely to be a solitary pursuit. I'd estimate the average is somewhere between 1 and 2 - let's take 1.5.

c) miles per gallon for the average car: Those of you who do drive probably know this better than I, but I think the average fuel efficiency of US cars is around 20 miles per gallon. Substituting for the subway, of course, would mean heavy City driving (indeed, it would be heavy City traffic jams if 3-4 million more cars came to New York each day!), so that's probably an overestimate; I'll adopt 15 miles per gallon.

IV. Getting the Answer

a) OK, we're ready to calculate:

4.9×10^6 riders per day $\times 31$ days/March $\times 4$ miles/rider $\times 1$ gallon/15 miles $= 4.0 \times 10^7$ gallons/March,

or 40 million gallons in one month! Note how all the other units canceled out and I was left with what I wanted: gallons/March. If I had mistakenly put in miles/gallon instead of gallons/mile in that term, the answer would have had the wrong units, cluing me in that I must have made a mistake. Note also that I quote my answer to one significant figure. The exact calculation yields 40,506,666.67, but it would be absurd to quote that for the answer since the input numbers are barely accurate to a factor of two. See Chapter 5 for more than you may want to know about significant figures.

b) Is this number plausible? Sure – there are almost 5 million people involved, and the idea of a person using 10 gallons of gas in a month isn't crazy (most use more, but not a

100 times more, nor 100 times less).

Sample problem 2: How about those meteors? Could your house really get hit by one?

I. Understanding the Problem

a) What we want to know: the odds of one's house being hit in a lifetime.

b) What we are given: 10^5 meteors big enough to reach the Earth's surface and penetrate a house's roof land each year.

c) Picture: probably not useful here. Although you might want to think about the relative size of a softball and a house roof. The latter is *much* bigger, so the softball will either hit the roof or not, and we only need to worry about the size of all targets (roofs). If we were dealing with 10 km-wide asteroids (like the one that wiped out the dinosaurs), it's much bigger than a roof, and we'd have to worry about its size as well. Fortunately, we only have to worry about ones like that every 100 million years, so we can probably ignore it in this instance.

II. Identifying Additional Information

a) What we need to know: My approach is to calculate the fraction of the Earth's surface covered by roofs to get the fraction of the time a roof is hit. This will require knowing the Earth's population and estimating the average roof area per person, as well as the size of the Earth. The Earth's population is a number it would be useful to remember: $\sim 6.5 \times 10^9$. The radius of the Earth is easy to look up, but is also something you should be able to make a rough estimate of (can you?): ~ 6400 km

b) Definitions: A basic one is what I mean by "probability". It is simply the number of outcomes of interest (in this case, a roof getting hit) divided by the total possible outcomes (anywhere on earth getting hit) – see Ch. 4 for an extended discussion of probability.

Also, we'll need to know the area of a sphere $= 4\pi R^2$, and the area of a roof = length x width

c) One peripheral issue might be "But what about all those people living in apartment buildings in cities? They have much less roof per person." This may be true, but 1) the large majority of the world's populations lives in rural areas, 2) the majority of people living in cities live in normal houses in the first world (ever been to Queens?) or in shanty towns in the third world (which don't exactly have high-rise condominiums), and, finally, 3) the estimate still doesn't change much (e.g., there are 43 apartments (roughly 100 people) living in my Columbia apartment building and it's roof area is about 900 m^2, so that would be 9 m^2 which is within a factor of two or so of my rough estimate).

d) Assumptions: I am assuming meteors land randomly all over the Earth; since I can't think of any reason they would have preferred landing places, this seems plausible.

III. Making Good Estimates

a) Roof area per person: Here it is important to recognize that most of the world's population does not live the American suburban lifestyle, so don't say 100 m^2 (which would be the answer for a four-person family in a 4000 ft^2 house). My estimate would be $4m^2$ and that may even be generous. My reasoning is that the large majority of the Earth's population does sleep indoors. The area needed for an adult to lie down is a minimum of $1m^2$, so it has to be bigger than this. On the other hand, it is almost certainly not ten times this area, because that would mean each person having a 10' by 10' living space, and that is certainly an overestimate, given that 53% of the world's population has in an income of less than $2 per day.

IV. Getting the Answer

a) Ready to calculate: $6.5 \times 10^9 \text{people} \times 4\text{m}^2/\text{person}/4\pi(6400\text{km} \times 10^3\text{m}/1\text{km})^2 = 5 \times 10^{-5}$ is the fraction of the Earth's surface covered by roofs. There are 100,000 chances per year, suggesting this happens several times a year somewhere in the world! In fact, it happened in 2003 just outside Chicago (*www.meterobs.org/maillist/msg*27439.*html*).

But what about *your* house? Just because someone wins the Lottery every day, it doesn't mean you will. US average household size is 2.61 and life expectancy is approaching 80 yrs. In addition, our average roof area per person is, as noted above, probably almost a factor of 10 above the world average. So if the world average is 5 times a year there would be 400 hits in your lifetime, and a US roof makes up $40\text{m}^2 \times 2.61\text{people/roof}/4\pi(6400\text{km} \times 10^3\text{m}/1\text{km})^2 = 2 \times 10^{-13}$ of the earth' surface $\times 10^5$per year $\times$ 80 yrs $= 1.6 \times 10^{-6}$ or a little over one a million, better than the odds in most state lotteries.

b) Is this plausible? Hard to say off-hand, but a modest amount of web research shows that there are several reported incidents of meteors crashing through roofs in the US in the past few decades, and since we make up only 4.5% of the world's population, that's about what one would expect if it happens several times a year somewhere in the world.

Here are some practice problems for you to try:

1. Roughly how many people in Columbia College (4050 students) have the same birthday as you? Approximately how many have the same day *and* date (i.e., were born on exactly the same day and year)?

2. McDonalds now claims they have sold over 300 billion burgers. If you were to spread these all out in layers over an area the size of Columbia's main campus, would you be up to your knees in burgers? The main campus is six blocks long and three blocks wide (see

above for translation of blocks to distance). Suppose you made a single stack how high would it be (ignoring the fact that the weight of the stack would squeeze all the grease out)?

3. Estimate the number of dump trucks full of toe nail clippings that are generated each week on Earth.

4. The diameter of an atom is about 10^{10} meters (they are all about the same size). Cells have an enormous range of sizes from tiny red blood cells to some neurons in your spinal cord that have extensions from a single cell more than two feet long(!). But let's take a more typical cell it is about 0.5 micron (= 1 micrometer = 1μ m = 10^{6} m) in diameter, and is roughly (good enough for this calculation) spherical. How many atoms are there is such a cell?

5. Calculate the approximate number of atoms in the period at the end of this sentence.

6. The Greenland Ice Cap extends roughly 2500 km from north to south and has an average width of 1000 km. It's mean thickness is about 2000 m, although in some places it is over 4200 m thick. If it were all to melt and run off into the world's oceans which cover about 70% of the planet today, how much would sea level rise? The Earth's radius is about 6400 km.

Just for fun, here's a back of the envelope gone wild that I was sent:

A Scientific Enquiry into Santa

1) No known species of reindeer can fly. *But* there are millions of species of living organisms yet to be classified, and while most of these are insects and bacteria, this does not *completely* rule out flying reindeer which only Santa has ever seen.

2) There are 2 billion children (persons under 18) in the world. *but* since Santa doesn't (appear) to handle the Muslim, Hindu, Jewish, and Buddhist children, that reduces the workload to 15% of the total 378 million according to Population Reference Bureau. At an average (census) rate of 3.5 children per household, that's 91.8 million homes. One presumes there's at least one good child in each.

3) Santa has 31 hours of Christmas to work with. This is due to the different time zones and the rotation of the earth, assuming he travels east to west (which seems logical). This works out to 822.6 visits/second. This is to say that for each Christian household with good children, Santa has .001 second to park, hop out of the sleigh, jump down the chimney, get back into the sleigh and move on to the next house. Assuming that each of these 91.8 million stops are evenly distributed around the earth (which, of course, we know to be false but for the purposes of our calculations we will accept), we are now talking about .78 miles/household, a total trip of 75.5 million miles; not counting stops to do what most

of us must do at least once every 31 hours, plus feeding, etc. So Santa's sleigh must be moving at 650 miles/second, 3,000 times the speed of sound. For purposes of comparison, the fastest man-made vehicle, the Ulysses space probe, moves at a poky 27.4 miles/second. A conventional reindeer can run, tops, 15 miles/hour (0.004 miles/second).

4) The payload on the sleigh adds another interesting element. Assuming that each child gets nothing more than a medium-sized Lego set (2 lb.), the sleigh is carrying 321,300 tons, not counting Santa, who is invariably described as overweight. On land, conventional reindeer can pull no more than 300 lb. Even granting that "flying reindeer" (see #1) could pull 10 times the normal amount, we cannot do the job with 8, or even 9 reindeer. We need 214,200. This increases the payload - not counting the weight of the sleigh - to 353,430 tons. This is four times the weight of the ocean-liner Queen Elizabeth II.

5) 353,000 tons traveling at 650 miles/second creates enormous air resistance. This will heat up the reindeer up in the same fashion as a spacecraft reentering the earth's atmosphere. The lead pair of reindeer will absorb 14.3 *quintillion* joules of energy. Per second. Each. In short, they will burst into flame almost instantaneously, exposing the reindeer behind them, and create deafening sonic booms in their wake. The entire reindeer team will be vaporized within .00426 seconds. Meanwhile, Santa will be subjected to inertial forces 17,500 times greater than gravity. A 250-lb Santa (seems ludicrously slim) would be pinned to the back of his sleigh by 4,315,015 lb. of force.

Here are the answers I got for the six practice problems. If you are within a factor of 2 or 3 on numbers 1, 4, 5, and 6, and within a factor of 10 or 20 on the others, you're on your way to mastering this habit of mind.

1. Roughly 11 with the same birth date and maybe 2 or 3 with an identical birthday.

2. The layer would be almost half a mile think – WAY over your knees, A single stack would reach roughly 4 million miles high, or sixteen times the distance to the Moon.

3. Ten or fifteen, I'd say.

4. About 100 billion atoms in a cell – same as the number of stars in the Milky Way, the number of galaxies in the visible Universe, and the number of neurons in your brain,

5. About 3×10^{18} atoms, or 3 billion billion – periods are a lot bigger than cells.

6. Since sea-level rise from melting ice caps is an interesting number for you to know, as well as being relevant to one of the Frontiers of Science units, I solve this one in detail here:

What we want to know: the amount of sea-level rise - in meters

What we are given: ice is 2500 km x 1000 km x2000 meters, the oceans area is 70% of the total area of Earth; the Earth's radius is 6400 km. Note we are also given the glacier's greatest thickness, but this will not be relevant.

Picture: if it is unclear how the volume of ice is going to spread out and produce a change in the average depth of the ocean change, a picture might help - with this giant ice cube melting into a thin surface layer of water, it should be clear that the volume of the ice is to be compared to the volume of the new ocean = area of ocean x added depth.

What we need to know: volume of ice, area of Earth's surface

Volume = length x width x height; area of a sphere $4\pi R^2$

Estimates: don't need any here - we have everything, unless you want to worry about the fact that ice is somewhat less dense than water and therefore takes up more space (it floats). But first, most of this ice is compressed under many tons of ice above and so isn't less dense; furthermore, this is only a 15% effect and most of our numbers aren't more accurate than this, so we can ignore this.

Ready to calculate:

Volume of ice = 2500km x 10^3m/kmx 1000km x 10^3m/km x 2000m = 5 x10^{15} m^3. Note how I converted all the units to meters (I could have converted to km, but since my answer is likely to be in meters I chose that). Now since we realized from the picture that the volume of water is just the area of the oceans's surface times the depth, all we need to do is divide this volume by the area. Ocean's surface = 4π x (6400km x 10^3m/km)^2 x 0.7 = 3.6 x 10^{14} m^2. Dividing this into the ice volume, that answer is about 14 meters or about 45 feet (which is not good news for most of the major cities of the world).